Partial Fractions

A **rational function** is a fraction in which both the numerator and denominator are polynomials. For example, $f(x)=\dfrac{4}{x-2}$, $g(x)=\dfrac{-3}{x+5}$, and $h(x)=\dfrac{x+26}{x^2+3x-10}$ are rational functions. You should already be quite familiar with performing algebraic operations with such fractions. For example, the sum of $f(x)$ and $g(x)$ is

$$f(x)+g(x)=\frac{4}{x-2}+\frac{-3}{x+5}=\frac{4(x+5)+-3(x-2)}{(x-2)(x+5)}=\frac{x+26}{x^2+3x-10}.$$

Thus the sum of $f(x)$ and $g(x)$ just happens to be $h(x)$.

In calculus and other advanced mathematics it is sometimes necessary to be able to go the other direction in the process just demonstrated; that is, given a more complicated rational function such as $h(x)$ we may need to determine simpler rational functions such as $f(x)$ and $g(x)$ whose sum is $h(x)$. By "simpler rational functions" we mean fractions that have lower degree denominators than the original. When such a process is carried out we say that we have **decomposed** the original rational function into **partial fractions**.

Before giving a complete discussion of how this can be accomplished, let's look at an example that will illustrate the reasoning behind the procedure.

Example 1: Decompose $\dfrac{-3x-23}{x^2-x-12}$ into partial fractions.

Solution: The denominator of the given fraction factors into $(x-4)(x+3)$. This product would be the common denominator of fractions that have $(x-4)$ and $(x+3)$ as their individual denominators. Thus it seems reasonable to expect that the given fraction might be written as the sum of two fractions, one with a denominator of $(x-4)$ and the other with a denominator of $(x+3)$. As yet we don't know the numerators so we simply call them A and B. Thus we have

$$\frac{-3x-23}{x^2-x-12}=\frac{A}{x-4}+\frac{B}{x+3}. \tag{1}$$

After determining the lowest common denominator (LCD) of the fractions on the right, we carry out their addition and get

$$\frac{-3x-23}{x^2-x-12}=\frac{A(x+3)+B(x-4)}{(x-4)(x+3)}=\frac{(A+B)x+(3A-4B)}{x^2-x-12}.$$

Notice that in the numerator of the final fraction we have grouped terms by powers of x; in other words, we have written the two x terms together and the two constant terms together.

Now since we want the fraction on the far right to equal the fraction on the far left for all values of x (for which the fractions are defined), it follows that the coefficients of corresponding powers of x in the numerators of those fractions must be equal. So $A+B$ must equal -3 and $3A-4B$ must equal -23. Thus we have the system of equations

$$\begin{cases} A+B=-3 \\ 3A-4B=-23 \end{cases}.$$

Systems of linear equations such as this can be solved by methods such as Substitution, Addition, and Gauss-Jordan Elimination. For this system we find that $A = -5$ and $B = 2$. Substituting these results into (1) we see then that

$$\frac{-3x-23}{x^2-x-12} = \frac{-5}{x-4} + \frac{2}{x+3}.$$

The preceding example illustrates the procedures used in decomposing a relatively simple rational function. Things can get considerably more complicated if, for example, the denominator of the fraction to be decomposed has quadratic factors or repeated factors. Before we discuss general procedures for decomposing a rational function, we need the following theorem. Its proof is left as Exercises 33 and 34.

Theorem 1: Let p, q, r, s, and t be real numbers. Any real polynomial can be factored into linear factors that have the form $px+q$ or quadratic factors that have the form rx^2+sx+t and no real zeros.

As a result of the preceding theorem we can see that the denominator of any rational function can be written as the product of factors of the form $(px+q)$and (rx^2+sx+t). If a factor occurs exactly m times we will say that it has **multiplicity** $\boldsymbol{m}$. Thus, in

$$\frac{5x^2-3x+1}{(2x-3)^3(x^2+1)^4},$$

the factor $(2x-3)$ has multiplicity 3 and the factor (x^2+1) has multiplicity 4.

With the assurance of Theorem 1 and the definition of a rational functions we are now able to state a general procedure for decomposing a rational function.

Procedure for Decomposing a Rational Function

To decompose the rational function $f(x) = \dfrac{g(x)}{h(x)}$, where the degree of $g(x)$ is less than the degree of $h(x)$:

1. Factor $h(x)$ into factors of the form $(px+q)^m$ and $(rx^2+sx+t)^n$, where m and n are the multiplicities of $(px+q)$ and (rx^2+sx+t), respectively.

2. For each power of a linear factor $(px+q)^m$, allow the decomposition to include terms of the form

$$\frac{A_1}{(px+q)}, \frac{A_2}{(px+q)^2}, \frac{A_3}{(px+q)^3}, \dots, \frac{A_m}{(px+q)^m},$$

where the A_j are real constants to be determined.

3. For each power of a quadratic factor $(rx^2+sx+t)^n$, allow the decomposition to include terms of the form

$$\frac{B_1x+C_1}{rx^2+sx+t}, \frac{B_2x+C_2}{(rx^2+sx+t)^2}, \frac{B_3x+C_3}{(rx^2+sx+t)^3}, \ldots, \frac{B_nx+C_n}{(rx^2+sx+t)^n},$$

where the B_j and C_j are real constants to be determined.

4. On the left side of an equation write the given rational function and on the right side indicate the sum of the terms that were to be included in the decomposition according to Steps 2 and 3.

5. Find the common denominator of the terms on the right and combine them. Group the terms in the numerator of this result by powers of x.

6. Equate the coefficients of like powers of x on the left and right sides of the resulting equation to get a system of equations involving the constants that appear in the numerators of the partial fractions.

7. Solve the system of equations obtained in Step 6.

8. Substitute the values determined in Step 7 for the corresponding constants in the numerators of the fractions on the right side of the equation written in Step 4.

Example 2: Decompose $\frac{-2x^2-5x-1}{(x+2)^3}$ into partial fractions.

Solution: Since the denominator of the given rational function is already factored, we go directly to Step 2 of the Procedure for Decomposing a Rational Function. Because the denominator consists of one factor and it has a multiplicity of 3, we are to allow three terms in the decomposition. According to Step 2 these terms are to have the first three powers of $(x+2)$ as denominators and constants as numerators. Because there are no quadratic factors in the given rational function, we skip Step 3 and do Step 4 next. Thus we write

$$\frac{-2x^2-5x-1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}. \qquad (2)$$

Taking Step 5 we find that the LCD for the terms on the right of (2) is $(x+2)^3$. Combining those terms yields

$$\frac{-2x^2-5x-1}{(x+2)^3} = \frac{A(x+2)^2+B(x+2)+C}{(x+2)^3} = \frac{A(x^2+4x+4)+Bx+2B+C}{(x+2)^3}$$

$$= \frac{Ax^2+(4A+B)x+(4A+2B+C)}{(x+2)^3}.$$

As directed in Step 6, we next equate the coefficients of the corresponding powers of x in the first and final fractions of the preceding equality and get the system

$$\begin{cases} A=-2 \\ 4A+B=-5 \\ 4A+2B+C=-1 \end{cases}.$$

For our seventh step we solve this system and find
$A=-2$, $B=3$, and $C=1$. Substituting these results into (A1.2) we have

$$\frac{-2x^2-5x-1}{(x+2)^3}=\frac{-2}{x+2}+\frac{3}{(x+2)^2}+\frac{1}{(x+2)^3}.$$

Example 3: Decompose $\dfrac{x^2+3x-15}{(x^2+9)(x^2+x+1)}$ into partial fractions.

Solution: Since the denominator is already factored and has no linear factors, we go directly to Step 3 of the procedure. Because each of the quadratic factors in the denominator is to the first power, our decomposition must allow for only two fractions–one having each of the two quadratic factors as its denominator. According to the directions in Step 3, we must allow the numerators of these fractions to be linear expressions. Thus by Step 4 we have

$$\frac{x^2+3x-15}{(x^2+9)(x^2+x+1)}=\frac{Ax+B}{x^2+9}+\frac{Cx+D}{x^2+x+1}. \tag{3}$$

Taking Step 5 we find that the product $(x^2+9)(x^2+x+1)$ is the LCD for the two fractions on the right of (3). Combining those two fractions yields

$$\begin{aligned}\frac{x^2+3x-15}{(x^2+9)(x^2+x+1)}&=\frac{(Ax+B)(x^2+x+1)+(Cx+D)(x^2+9)}{(x^2+9)(x^2+x+1)}\\ &=\frac{Ax^3+Ax^2+Ax+Bx^2+Bx+B+Cx^3+9Cx+Dx^2+9D}{(x^2+9)(x^2+x+1)}\\ &=\frac{(A+C)x^3+(A+B+D)x^2+(A+B+9C)x+(B+9D)}{(x^2+9)(x^2+x+1)}.\end{aligned}$$

As instructed in Step 6, we equate the coefficients of like powers of x on the far left and right sides of the preceding result and obtain the system of equations.

$$\begin{cases}A+C=0\\ A+B+D=1\\ A+B+9C=3\\ B+9D=-15\end{cases}.$$

Doing Step 7 we solve this system and find that $A=0$, $B=3$, $C=0$, and $D=-2$. Finally taking Step 8 we substitute these values into (3) to produce the desired decomposition

$$\frac{x^2+3x-15}{(x^2+9)(x^2+x+1)}=\frac{3}{x^2+9}-\frac{2}{x^2+x+1}.$$

Example 4: Decompose $\dfrac{x^3+3x^2-12x+36}{x^4-16}$ into partial fractions.

Solution: First we factor the denominator into $(x-2)(x+2)(x^2+4)$. Each of the three factors is of multiplicity 1 so we must allow for three partial fractions in the decomposition, one with a denominator of $(x-2)$, one with a denominator of $(x+2)$, and one with a denominator of (x^2+4). For the numerators that go over the linear denominators we need allow only constants while for the numerator that goes over the quadratic denominator we must allow a linear expression. Thus we write

$$\frac{x^3+3x^2-12x+36}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}. \tag{4}$$

Combining the fractions on the right yields

$$\begin{aligned}
&\frac{x^3+3x^2-12x+36}{(x-2)(x+2)(x^2+4)} \\
&= \frac{A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x-2)(x+2)}{(x-2)(x+2)(x^2+4)} \\
&= \frac{A(x^3+4x+2x^2+8)+B(x^3+4x-2x^2-8)+Cx^3-4Cx+Dx^2-4D}{(x-2)(x+2)(x^2+4)} \\
&= \frac{(A+B+C)x^3+(2A-2B+D)x^2+(4A+4B-4C)x+(8A-8B-4D)}{(x-2)(x+2)(x^2+4)}.
\end{aligned}$$

Equating the coefficients of like powers of x in the numerators of the first and final fractions of this equality produces the system of equations

$$\begin{cases} A+B+C=1 \\ 2A-2B+D=3 \\ 4A+4B-4C=-12 \\ 8A-8B-4D=36 \end{cases}.$$

Solving this system we find that $A=1$, $B=-2$, $C=2$, and $D=-3$. Substituting these values into (4) yields

$$\frac{x^3+3x^2-12x+36}{(x-2)(x+2)(x^2+4)} = \frac{1}{x-2} - \frac{2}{x+2} + \frac{2x-3}{x^2+4}.$$

In the procedure we have been using to decompose rational functions into partial fractions, it is necessary that the degree of the numerator of the original rational function be less that the degree of its denominator. If this is not the case then polynomial division is used to get a quotient plus remainder. The remainder thus obtained is a rational function in which the degree of the numerator is less than the degree of the denominator and that therefore can be decomposed into partial fractions by the procedure we have been using. This treatment of rational functions produces results for which convenient calculus and other mathematical formulas are available whereas no such formulas are generally available for the original rational function.

Example 5: Write $\dfrac{x^4-4x^3+3x^2+x+3}{x^3-x^2+x-1}$ as the sum of a polynomial and partial fractions.

Solution: We first use polynomial division to divide the numerator of the given rational function by its denominator. This yields

$$\frac{x^4-4x^3+3x^2+x+3}{x^3-x^2+x-1} = x-3 + \frac{-x^2+5x}{x^3-x^2+x-1} \tag{5}$$

Now we proceed to decompose the remainder term into partial fractions. We can factor its denominator by grouping to get

$x^3 - x^2 + x - 1 = x^2(x-1) + 1(x-1) = (x^2+1)(x-1)$. Thus in our decomposition we allow one term with (x^2+1) as its denominator and a linear expression as its numerator, and another term with $(x-1)$ as its denominator and a constant as its numerator. In other words, we let

$$\frac{-x^2+5x}{x^3-x^2+x-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}. \tag{6}$$

Combining the fractions on the right we find

$$\frac{-x^2+5x}{x^3-x^2+x-1} = \frac{(Ax+B)(x-1)+C(x^2+1)}{(x^2+1)(x-1)} = \frac{Ax^2 - Ax + Bx - B + Cx^2 + C}{(x^2+1)(x-1)}$$

$$= \frac{(A+C)x^2 + (-A+B)x + (-B+C)}{(x^2+1)(x-1)}.$$

Equating coefficients of corresponding powers of x in the first and final numerators we get the system of equations

$$\begin{cases} A+C=-1 \\ -A+B=5 \\ -B+C=0 \end{cases},$$

which has $A=-3$, $B=2$, and $C=2$ as its solution. Substituting these results into (6) we have

$$\frac{-x^2+5x}{x^3-x^2+x-1} = \frac{-3x+2}{x^2+1} + \frac{2}{x-1}.$$

Substituting this decomposition into (5) we obtain our final result

$$\frac{x^4-4x^3+3x^2+x+3}{x^3-x^2+x-1} = x-3+\frac{-3x+2}{x^2+1}+\frac{2}{x-1}.$$

Partial Fraction Exercises

In Exercises 1-26, decompose the given rational function into partial fractions.

1. $\dfrac{-x+13}{x^2-x-6}$

2. $\dfrac{-2x+23}{x^2-3x-4}$

3. $\dfrac{9x-7}{2x^2-3x+1}$

4. $\dfrac{x+11}{2x^2-x-10}$

5. $\dfrac{-4}{x^2-4}$

6. $\dfrac{4x+12}{4x^2-9}$

7. $\dfrac{3x+10}{(x+4)^2}$

8. $\dfrac{5x-7}{(x-2)^2}$

Form of Rational Function	Form of the Partial Fraction
$\frac{p(x)+q}{(x-a)(x-b)} a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{p(x)+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ Where, $x^2 + bx + c$ cannot be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$ A, B, C are real numbers that are to be determined.